76. The spin rate of a golf ball hit with a 9 iron has been measured at 9100 rpm for a 120-compression ball and at 10,000 rpm for a 60-compression ball. Most golfers use 90-compression balls. If the spin rate is a linear function of compression, find the spin rate for a 90-compression ball. Professional golfers often use 100-compression balls. Estimate the spin rate of a 100-compression ball.

77. The chirping rate of a cricket depends on the temperature. A species of tree cricket chirps 160 times per minute at 79°F and 100 times per minute at 64°F. Find a linear function relating temperature to chirping rate.

78. When describing how to measure temperature by counting cricket chirps, most guides suggest that you count the number of chirps in a 15-second time period. Use exercise 77 to explain why this is a convenient period of time.

79. A person has played a computer game many times. The statistics show that she has won 415 times and lost 120 times, and the winning percentage is listed as 78%. How many times in a row must she win to raise the reported winning percentage to 80%?

EXPLORATORY EXERCISES

1. Suppose you have a machine that will proportionally enlarge a photograph. For example, it could enlarge a 4×6 photograph to 8×12 by doubling the width and height. You could make an 8×10 picture by cropping 1 inch off each side. Explain how you would enlarge a $3\frac{1}{2} \times 5$ picture to an 8×10. A friend returns from Scotland with a $3\frac{1}{2} \times 5$ picture showing the Loch Ness monster in the outer $\frac{1}{4}''$ on the right. If you use your procedure to make an 8×10 enlargement, does Nessie make the cut?

2. Solve the equation $|x-2| + |x-3| = 1$. (Hint: It's an unusual solution, in that it's more than just a couple of numbers.) Then, solve the equation $\sqrt{x+3-4\sqrt{x-1}} + \sqrt{x+8-6\sqrt{x-1}} = 1$. (Hint: If you make the correct substitution, you can use your solution to the previous equation.)

0.2 GRAPHING CALCULATORS AND COMPUTER ALGEBRA SYSTEMS

The relationships between functions and their graphs are central topics in calculus. Graphing calculators and user-friendly computer software allow you to explore these relationships for a much wider variety of functions than you could with pencil and paper alone. This section presents a general framework for using technology to explore the graphs of functions.

Recall that the graphs of linear functions are straight lines and the graphs of quadratic polynomials are parabolas. One of the goals of this section is for you to become more familiar with the graphs of other functions. The best way to become familiar is through experience, by working example after example.

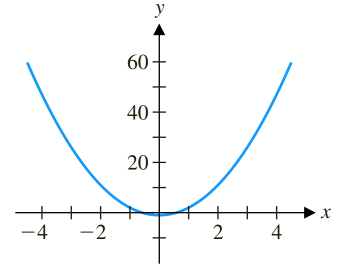

FIGURE 0.26a
$y = 3x^2 - 1$

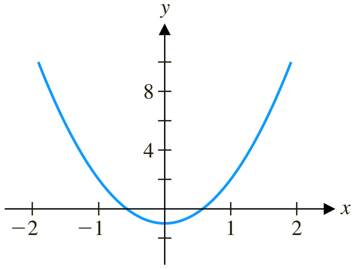

FIGURE 0.26b
$y = 3x^2 - 1$

EXAMPLE 2.1 Generating a Calculator Graph

Use your calculator or computer to sketch a graph of $f(x) = 3x^2 - 1$.

Solution You should get an initial graph that looks something like that in Figure 0.26a. This is simply a parabola opening upward. A graph is often used to search for important points, such as x-intercepts, y-intercepts or peaks and troughs. In this case, we could see these points better if we zoom in, that is, display a smaller range of x- and y-values than the technology has initially chosen for us. The graph in Figure 0.26b shows x-values from $x = -2$ to $x = 2$ and y-values from $y = -2$ to $y = 10$.

You can see more clearly in Figure 0.26b that the parabola bottoms out roughly at the point $(0, -1)$ and crosses the x-axis at approximately $x = -0.5$ and $x = 0.5$. You can make this more precise by doing some algebra. Recall that an x-intercept is a point where $y = 0$ or $f(x) = 0$. Solving $3x^2 - 1 = 0$ gives $3x^2 = 1$ or $x^2 = \frac{1}{3}$, so that $x = \pm\sqrt{\frac{1}{3}} \approx \pm 0.57735$. ■

Notice that in example 2.1, the graph suggested approximate values for the two x-intercepts, but we needed the algebra to find the values exactly. We then used those values to obtain a view of the graph that highlighted the features that we wanted.

Before investigating other graphs, we should say a few words about what a computer- or calculator-generated graph really is. Although we call them graphs, what the computer actually does is light up some tiny screen elements called **pixels.** If the pixels are small enough, the image appears to be a continuous curve or graph.

By **graphing window,** we mean the rectangle defined by the range of x- and y-values displayed. The graphing window can dramatically affect the look of a graph. For example, suppose the x's run from $x = -2$ to $x = 2$. If the computer or calculator screen is wide enough for 400 columns of pixels from left to right, then points will be displayed for $x = -2, x = -1.99, x = -1.98, \ldots$. If there is an interesting feature of this function located between $x = -1.99$ and $x = -1.98$, you will not see it unless you zoom in some. In this case, zooming in would reduce the difference between adjacent x's. Similarly, suppose that the y's run from $y = 0$ to $y = 3$ and that there are 600 rows of pixels from top to bottom. Then, there will be pixels corresponding to $y = 0, y = 0.005, y = 0.01, \ldots$. Now, suppose that $f(-2) = 0.0049$ and $f(-1.99) = 0.0051$. Before points are plotted, function values are rounded to the nearest y-value, in this case 0.005. You won't be able to see any difference in the y-values of these points. If the actual difference is important, you will have to zoom in some to see it.

REMARK 2.1

Most calculators and computer drawing packages use one of the following two schemes for defining the graphing window for a given function.

- **Fixed graphing window:** Most calculators follow this method. Graphs are plotted in a preselected range of x- and y-values, unless you specify otherwise. For example, the Texas Instruments graphing calculators' default graphing window plots points in the rectangle defined by $-10 \le x \le 10$ and $-10 \le y \le 10$.
- **Automatic graphing window:** Most computer drawing packages and some calculators use this method. Graphs are plotted for a preselected range of x-values and the computer calculates the range of y-values so that all of the calculated points will fit in the window.

Get to know how your calculator or computer software operates, and use it routinely as you progress through this course. You should always be able to reproduce the computer-generated graphs used in this text by adjusting your graphing window appropriately.

Graphs are drawn to provide visual displays of the significant features of a function. What qualifies as *significant* will vary from problem to problem, but often the x- and y-intercepts and points known as **extrema** are of interest. The function value $f(M)$ is called a **local maximum** of the function f if $f(M) \ge f(x)$ for all x's "nearby" $x = M$.

Similarly, the function value $f(m)$ is a **local minimum** of the function f if $f(m) \le f(x)$ for all x's "nearby" $x = m$. A **local extremum** is a function value that is either a local maximum or local minimum. Whenever possible, you should produce graphs that show all intercepts and extrema.

REMARK 2.2

To be precise, $f(M)$ is a local maximum of f if there exist numbers a and b with $a < M < b$ such that $f(M) \ge f(x)$ for all x such that $a < x < b$.

EXAMPLE 2.2 Sketching a Graph

Sketch a graph of $f(x) = x^3 + 4x^2 - 5x - 1$ showing all intercepts and extrema.

Solution Depending on your calculator or computer software, you may initially get a graph that looks like one of those in Figure 0.27a or 0.27b.

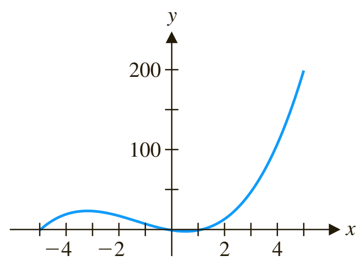

FIGURE 0.27a
$y = x^3 + 4x^2 - 5x - 1$

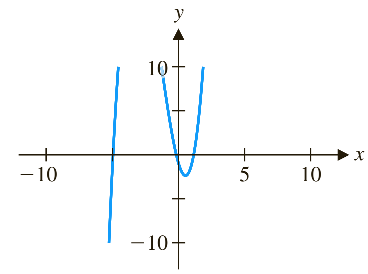

FIGURE 0.27b
$y = x^3 + 4x^2 - 5x - 1$

Neither graph is completely satisfactory, although both should give you the idea of a graph that (reading left to right) rises to a local maximum near $x = -3$, drops to a local minimum near $x = 1$, and then rises again. To get a better graph, notice the scales on the x- and y-axes. The graphing window for Figure 0.27a is the rectangle defined by $-5 \le x \le 5$ and $-6 \le y \le 203$. The graphing window for Figure 0.27b is defined by the rectangle $-10 \le x \le 10$ and $-10 \le y \le 10$. From either graph, it appears that we need to show y-values larger than 10, but not nearly as large as 203, to see the local maximum. Since all of the significant features appear to lie between $x = -6$ and $x = 6$, one choice for a better window is $-5 \le x \le 5$ and $-6 \le y \le 30$, as seen in Figure 0.28. There, you can clearly see the three x-intercepts, the local maximum and the local minimum. ■

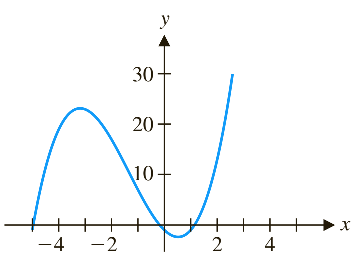

FIGURE 0.28
$y = x^3 + 4x^2 - 5x - 1$

The graph in example 2.2 was produced by a process of trial and error with thoughtful corrections. You are unlikely to get a perfect picture on the first try. However, you can enlarge the graphing window (i.e., *zoom out*) if you need to see more, or shrink the graphing window (i.e., *zoom in*) if the details are hard to see. You should get comfortable enough with your technology that this revision process is routine (and even fun!).

In the exercises, you will be asked to graph a variety of functions and discuss the shapes of the graphs of polynomials of different degrees. Having some knowledge of the general shapes will help you decide whether you have found an acceptable graph. To get you started, we now summarize the different shapes of linear, quadratic and cubic polynomials. Of course, the graphs of linear functions $[f(x) = a_1x + a_0]$ are simply straight lines of slope a_1. Two possibilities are shown in Figures 0.29a and 0.29b.

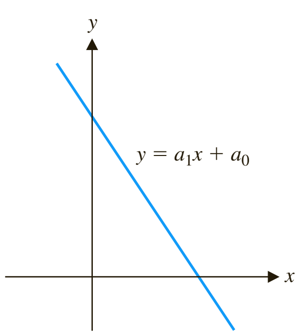

FIGURE 0.29a
Line, $a_1 < 0$

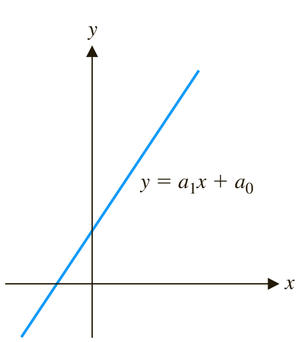

FIGURE 0.29b
Line, $a_1 > 0$

The graphs of quadratic polynomials $[f(x) = a_2x^2 + a_1x + a_0]$ are parabolas. The parabola opens upward if $a_2 > 0$ and opens downward if $a_2 < 0$. We show typical parabolas in Figures 0.30a and 0.30b.

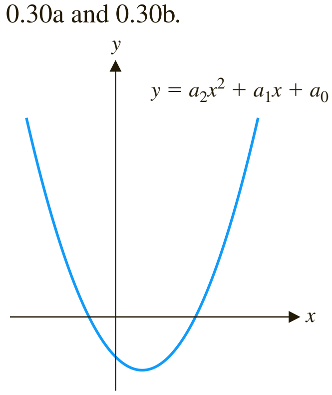

FIGURE 0.30a
Parabola, $a_2 > 0$

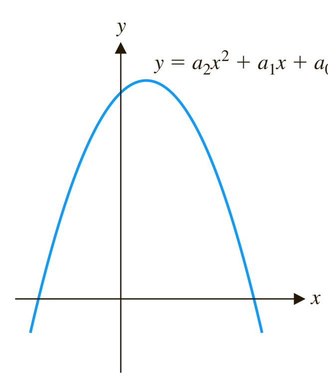

FIGURE 0.30b
Parabola, $a_2 < 0$

The graphs of cubic functions $[f(x) = a_3x^3 + a_2x^2 + a_1x + a_0]$ are somewhat S-shaped. Reading from left to right, the function begins negative and ends positive if $a_3 > 0$, and begins positive and ends negative if $a_3 < 0$, as indicated in Figures 0.31a and 0.31b, respectively.

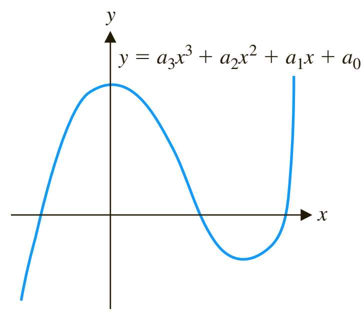

FIGURE 0.31a
Cubic: one max, min, $a_3 > 0$

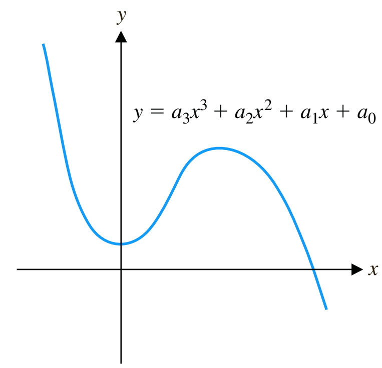

FIGURE 0.31b
Cubic: one max, min, $a_3 < 0$

Some cubics have one local maximum and one local minimum, as do those in Figures 0.31a and 0.31b. Many curves (including all cubics) have what's called an **inflection point,** where the curve changes its shape (from being bent upward, to being bent downward, or vice versa), as indicated in Figures 0.32a and 0.32b.

You can already use your knowledge of the general shapes of certain functions to see how to adjust the graphing window, as in example 2.3.

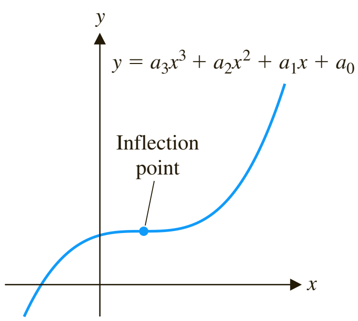

FIGURE 0.32a
Cubic: no max or min, $a_3 > 0$

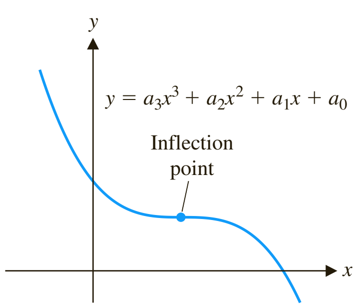

FIGURE 0.32b
Cubic: no max or min, $a_3 < 0$

EXAMPLE 2.3 Sketching the Graph of a Cubic Polynomial

Sketch a graph of the cubic polynomial $f(x) = x^3 - 20x^2 - x + 20$.

Solution Your initial graph probably looks like Figure 0.33a or 0.33b.

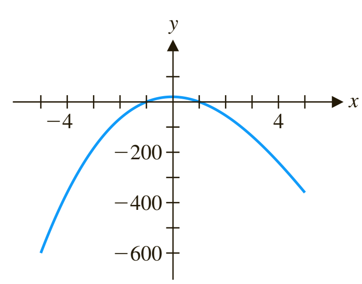

FIGURE 0.33a
$f(x) = x^3 - 20x^2 - x + 20$

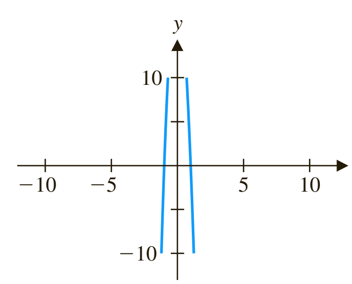

FIGURE 0.33b
$f(x) = x^3 - 20x^2 - x + 20$

However, you should recognize that neither of these graphs looks like a cubic; they look more like parabolas. To see the S-shape behavior in the graph, we need to consider a larger range of x-values. To determine how much larger, we need some of the concepts

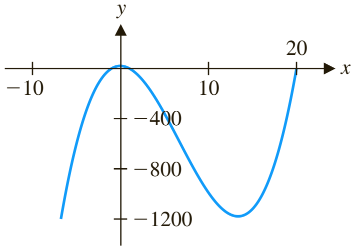

FIGURE 0.33c
$f(x) = x^3 - 20x^2 - x + 20$

of calculus. For the moment, we use trial and error, until the graph resembles the shape of a cubic. You should recognize the characteristic shape of a cubic in Figure 0.33c. Although we now see more of the big picture (often referred to as the **global** behavior of the function), we have lost some of the details (such as the x-intercepts), which we could clearly see in Figures 0.33a and 0.33b (often referred to as the **local** behavior of the function). ■

Rational functions have some properties not found in polynomials, as we see in examples 2.4, 2.5 and 2.6.

EXAMPLE 2.4 Sketching the Graph of a Rational Function

Sketch a graph of $f(x) = \dfrac{x-1}{x-2}$ and describe the behavior of the graph near $x = 2$.

Solution Your initial graph should look something like Figure 0.34a or 0.34b. From either graph, it should be clear that something unusual is happening near $x = 2$. Zooming in closer to $x = 2$ should yield a graph like that in Figure 0.35.

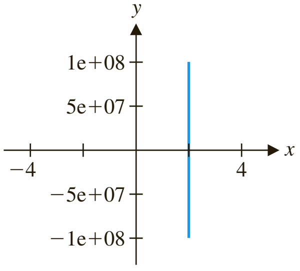

FIGURE 0.34a
$y = \dfrac{x-1}{x-2}$

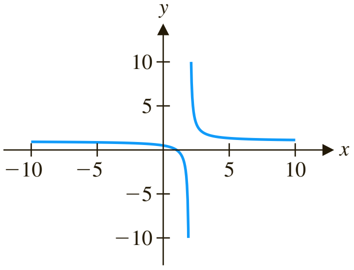

FIGURE 0.34b
$y = \dfrac{x-1}{x-2}$

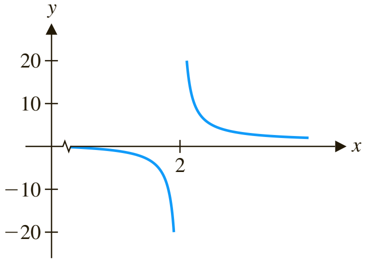

FIGURE 0.35
$y = \dfrac{x-1}{x-2}$

In Figure 0.35, it appears that as x increases up to 2, the function values get more and more negative, while as x decreases down to 2, the function values get more and more positive. This is also observed in the following table of function values.

x	$f(x)$
1.8	−4
1.9	−9
1.99	−99
1.999	−999
1.9999	−9999

x	$f(x)$
2.2	6
2.1	11
2.01	101
2.001	1001
2.0001	10,001

Note that at $x = 2$, $f(x)$ is undefined. However, as x approaches 2 from the left, the graph veers down sharply. In this case, we say that $f(x)$ tends to $-\infty$. Likewise, as x approaches 2 from the right, the graph rises sharply. Here, we say that $f(x)$ tends to ∞ and there is a **vertical asymptote** at $x = 2$. (We'll define this more carefully in Chapter 1.) It is common to draw a vertical dashed line at $x = 2$ to indicate this (see Figure 0.36). Since $f(2)$ is undefined, there is no point plotted at $x = 2$. ■

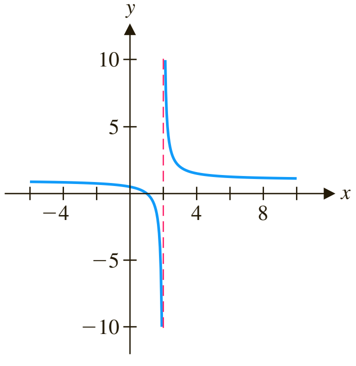

FIGURE 0.36
Vertical asymptote

Many rational functions have vertical asymptotes. Notice that there is no point plotted on the vertical asymptote since the function is undefined at such an x-value (due to the division by zero when that value of x is substituted in). Given a rational function, you can locate possible vertical asymptotes by finding where the denominator is zero. It turns out that if the numerator is not zero at that point, there is a vertical asymptote at that point.

EXAMPLE 2.5 A Graph with Several Vertical Asymptotes

Find all vertical asymptotes for $f(x) = \dfrac{x-1}{x^2-5x+6}$.

Solution Note that the denominator factors as

$$x^2 - 5x + 6 = (x-2)(x-3),$$

so that the only possible locations for vertical asymptotes are $x = 2$ and $x = 3$. Since neither x-value makes the numerator $(x - 1)$ equal to zero, there are vertical asymptotes at both $x = 2$ and $x = 3$. A computer-generated graph gives little indication of how the function behaves near the asymptotes. (See Figure 0.37a and note the scale on the y-axis.)

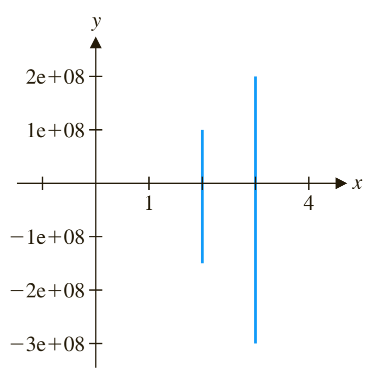

FIGURE 0.37a
$y = \dfrac{x-1}{x^2-5x+6}$

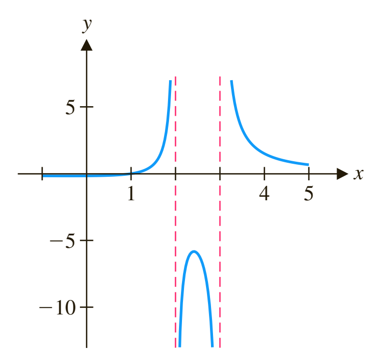

FIGURE 0.37b
$y = \dfrac{x-1}{x^2-5x+6}$

We can improve the graph by zooming in in both the x- and y-directions. Figure 0.37b shows a graph of the same function using the graphing window defined by the rectangle $-1 \le x \le 5$ and $-13 \le y \le 7$. This graph clearly shows the vertical asymptotes at $x = 2$ and $x = 3$. ■

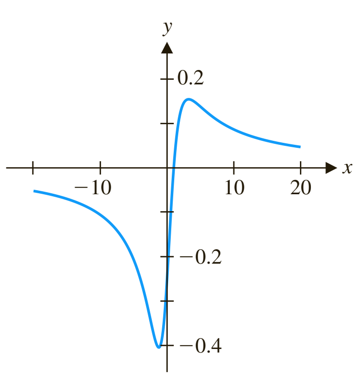

FIGURE 0.38
$y = \dfrac{x-1}{x^2+4}$

As we see in example 2.6, not all rational functions have vertical asymptotes.

EXAMPLE 2.6 A Rational Function with No Vertical Asymptotes

Find all vertical asymptotes of $\dfrac{x-1}{x^2+4}$.

Solution Notice that $x^2 + 4 = 0$ has no (real) solutions, since $x^2 + 4 > 0$ for all real numbers, x. So, there are no vertical asymptotes. The graph in Figure 0.38 is consistent with this observation. ■

Graphs are useful for finding approximate solutions of difficult equations, as we see in examples 2.7 and 2.8.

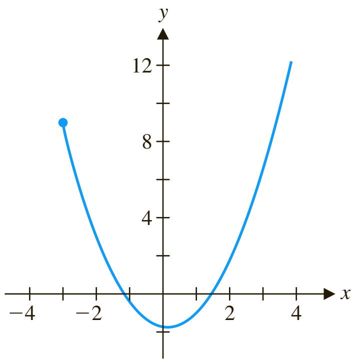

FIGURE 0.39a
$y = x^2 - \sqrt{x+3}$

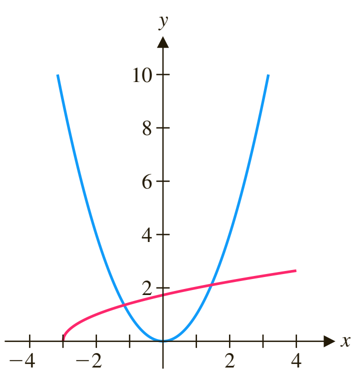

FIGURE 0.39b
$y = x^2$ and $y = \sqrt{x+3}$

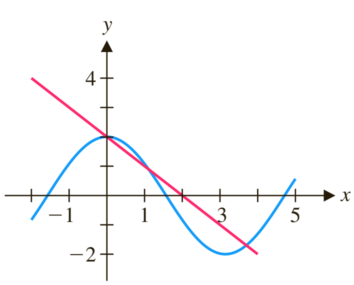

FIGURE 0.40
$y = 2\cos x$ and $y = 2 - x$

EXAMPLE 2.7 Finding Zeros Approximately

Find approximate solutions of the equation $x^2 = \sqrt{x+3}$.

Solution You could rewrite this equation as $x^2 - \sqrt{x+3} = 0$ and then look for zeros in the graph of $f(x) = x^2 - \sqrt{x+3}$, seen in Figure 0.39a. Note that two zeros are clearly indicated: one near -1, the other near 1.5. However, since you know very little of the nature of the function $x^2 - \sqrt{x+3}$, you cannot say whether or not there are any other zeros, ones that don't show up in the window seen in Figure 0.39a. On the other hand, if you graph the two functions on either side of the equation on the same set of axes, as in Figure 0.39b, you can clearly see two points where the graphs intersect (corresponding to the two zeros seen in Figure 0.39a). Further, since you know the general shapes of both of the graphs, you can infer from Figure 0.39b that there are no other intersections (i.e., there are no other zeros of f). This is important information that you cannot obtain from Figure 0.39a. Now that you know how many solutions there are, you need to estimate their values. One method is to zoom in on the zeros graphically. We leave it as an exercise to verify that the zeros are approximately $x = 1.4$ and $x = -1.2$. If your calculator or computer algebra system has a solve command, you can use it to quickly obtain an accurate approximation. In this case, we get $x \approx 1.452626878$ and $x \approx -1.164035140$. ■

When using the solve command on your calculator or computer algebra system, be sure to check that the solutions make sense. If the results don't match what you've seen in your preliminary sketches and zooms, beware! Even high-tech equation solvers make mistakes occasionally.

EXAMPLE 2.8 Finding Intersections by Calculator: An Oversight

Find all points of intersection of the graphs of $y = 2\cos x$ and $y = 2 - x$.

Solution Notice that the intersections correspond to solutions of the equation $2\cos x = 2 - x$. Using the solve command on the TI-92 graphing calculator, we found intersections at $x \approx 3.69815$ and $x = 0$. So, what's the problem? A sketch of the graphs of $y = 2 - x$ and $y = 2\cos x$ (we discuss this function in the next section) clearly shows three intersections (see Figure 0.40).

The middle solution, $x \approx 1.10914$, was somehow passed over by the calculator's solve routine. The lesson here is to use graphical evidence to support your solutions, especially when using software and/or functions with which you are less than completely familiar. ■

You need to look skeptically at the answers provided by your calculator's solver program. While such solvers provide a quick means of approximating solutions of equations, these programs will sometimes return incorrect answers, as we illustrate with example 2.9. So, how do you know if your solver is giving you an accurate answer or one that's incorrect? The only answer to this is that you must carefully test your calculator's solution, by separately calculating both sides of the equation (*by hand*) at the calculated solution.

EXAMPLE 2.9 Solving an Equation by Calculator: An Erroneous Answer

Use your calculator's solver program to solve the equation $x + \frac{1}{x} = \frac{1}{x}$.

Solution Certainly, you don't need a calculator to solve this equation, but consider what happens when you use one. Most calculators report a solution that is very close to zero, while others report that the solution is $x = 0$. Not only are these answers incorrect, but the given equation has *no* solution, as follows. First, notice that the equation makes sense only when $x \neq 0$. Subtracting $\frac{1}{x}$ from both sides of the equation leaves us with $x = 0$, which can't possibly be a solution, since it does not satisfy the original equation. Notice further that, if your calculator returns the approximate solution $x = 1 \times 10^{-7}$ and you use your calculator to compute the values on both sides of the equation, the calculator will compute

$$x + \frac{1}{x} = 1 \times 10^{-7} + 1 \times 10^{7},$$

which it approximates as $1 \times 10^{7} = \frac{1}{x}$, since calculators carry only a finite number of digits. In other words, although

$$1 \times 10^{-7} + 1 \times 10^{7} \neq 1 \times 10^{7},$$

your calculator treats these numbers as the same and so incorrectly reports that the equation is satisfied. The moral of this story is to be an intelligent user of technology and don't blindly accept everything a calculator tells you. ■

We want to emphasize again that graphing should be the first step in the equation-solving process. A good graph will show you how many solutions to expect, as well as give their approximate locations. Whenever possible, you should factor or use the quadratic formula to get exact solutions. When this is impossible, approximate the solutions by zooming in on them graphically or by using your calculator's solve command. *Always* compare your results to a graph to see if there's anything you've missed.

EXERCISES 0.2

WRITING EXERCISES

1. Explain why there is a significant difference among Figures 0.33a, 0.33b and 0.33c.

2. In Figure 0.36, the graph approaches the lower portion of the vertical asymptote from the left, whereas the graph approaches the upper portion of the vertical asymptote from the right. Use the table of function values found in example 2.4 to explain how to determine whether a graph approaches a vertical asymptote by dropping down or rising up.

3. In the text, we discussed the difference between graphing with a fixed window versus an automatic window. Discuss the advantages and disadvantages of each. (Hint: Consider the case of a first graph of a function you know nothing about and the case of hoping to see the important details of a graph for which you know the general shape.)

4. Examine the graph of $y = \frac{x^3 + 1}{x}$ with each of the following graphing windows: (a) $-10 \leq x \leq 10$, (b) $-1000 \leq x \leq 1000$. Explain why the graph in (b) doesn't show the details that the graph in (a) does.

In exercises 1–30, sketch a graph of the function showing all extrema, intercepts and asymptotes.

1. $f(x) = x^2 - 1$
2. $f(x) = 3 - x^2$
3. $f(x) = x^2 + 2x + 8$
4. $f(x) = x^2 - 20x + 11$
5. $f(x) = x^3 + 1$
6. $f(x) = 10 - x^3$
7. $f(x) = x^3 + 2x - 1$
8. $f(x) = x^3 - 3x + 1$

9. $f(x) = x^4 - 1$

10. $f(x) = 2 - x^4$

11. $f(x) = x^4 + 2x - 1$

12. $f(x) = x^4 - 6x^2 + 3$

13. $f(x) = x^5 + 2$

14. $f(x) = 12 - x^5$

15. $f(x) = x^5 - 8x^3 + 20x - 1$

16. $f(x) = x^5 + 5x^4 + 2x^3 + 1$

17. $f(x) = \dfrac{3}{x-1}$

18. $f(x) = \dfrac{4}{x+2}$

19. $f(x) = \dfrac{3x}{x-1}$

20. $f(x) = \dfrac{4x}{x+2}$

21. $f(x) = \dfrac{3x^2}{x-1}$

22. $f(x) = \dfrac{4x^2}{x+2}$

23. $f(x) = \dfrac{2}{x^2-4}$

24. $f(x) = \dfrac{6}{x^2-9}$

25. $f(x) = \dfrac{3}{x^2+4}$

26. $f(x) = \dfrac{6}{x^2+9}$

27. $f(x) = \dfrac{x+2}{x^2+x-6}$

28. $f(x) = \dfrac{x-1}{x^2+4x+3}$

29. $f(x) = \dfrac{3x}{\sqrt{x^2+4}}$

30. $f(x) = \dfrac{2x}{\sqrt{x^2+1}}$

In exercises 31–38, find all vertical asymptotes.

31. $f(x) = \dfrac{3x}{x^2-4}$

32. $f(x) = \dfrac{x+4}{x^2-9}$

33. $f(x) = \dfrac{4x}{x^2+3x-10}$

34. $f(x) = \dfrac{x+2}{x^2-2x-15}$

35. $f(x) = \dfrac{4x}{x^2+4}$

36. $f(x) = \dfrac{3x}{\sqrt{x^2-9}}$

37. $f(x) = \dfrac{x^2+1}{x^3+3x^2+2x}$

38. $f(x) = \dfrac{3x}{x^4-1}$

In exercises 39–42, a standard graphing window will not reveal all of the important details of the graph. Adjust the graphing window to find the missing details.

39. $f(x) = \frac{1}{3}x^3 - \frac{1}{400}x$

40. $f(x) = x^4 - 11x^3 + 5x - 2$

41. $f(x) = x\sqrt{144 - x^2}$

42. $f(x) = \frac{1}{5}x^5 - \frac{7}{8}x^4 + \frac{1}{3}x^3 + \frac{7}{2}x^2 - 6x$

In exercises 43–48, adjust the graphing window to identify all vertical asympotes.

43. $f(x) = \dfrac{3}{x-1}$

44. $f(x) = \dfrac{4x}{x^2-1}$

45. $f(x) = \dfrac{3x^2}{x^2-1}$

46. $f(x) = \dfrac{2x}{x+4}$

47. $f(x) = \dfrac{x^2-1}{\sqrt{x^4+x}}$

48. $f(x) = \dfrac{2x}{\sqrt{x^2+x}}$

In exercises 49–56, determine the number of (real) solutions. Solve for the intersection points exactly if possible and estimate the points if necessary.

49. $\sqrt{x-1} = x^2 - 1$

50. $\sqrt{x^2+4} = x^2 + 2$

51. $x^3 - 3x^2 = 1 - 3x$

52. $x^3 + 1 = -3x^2 - 3x$

53. $(x^2-1)^{2/3} = 2x + 1$

54. $(x+1)^{2/3} = 2 - x$

55. $\cos x = x^2 - 1$

56. $\sin x = x^2 + 1$

In exercises 57–62, use a graphing calculator or computer graphing utility to estimate all zeros.

57. $f(x) = x^3 - 3x + 1$

58. $f(x) = x^3 - 4x^2 + 2$

59. $f(x) = x^4 - 3x^3 - x + 1$

60. $f(x) = x^4 - 2x + 1$

61. $f(x) = x^4 - 7x^3 - 15x^2 - 10x - 1410$

62. $f(x) = x^6 - 4x^4 + 2x^3 - 8x - 2$

63. Graph $y = x^2$ in the graphing window $-10 \le x \le 10$, $-10 \le y \le 10$, without drawing the x- and y-axes. Adjust the graphing window for $y = 2(x-1)^2 + 3$ so that (without the axes showing) the graph looks identical to that of $y = x^2$.

64. Graph $y = x^2$ in the graphing window $-10 \le x \le 10$, $-10 \le y \le 10$. Separately graph $y = x^4$ with the same graphing window. Compare and contrast the graphs. Then graph the two functions on the same axes and carefully examine the differences in the intervals $-1 < x < 1$ and $x > 1$.

65. In this exercise, you will find an equation describing all points equidistant from the x-axis and the point $(0, 2)$. First, see if you can sketch a picture of what this curve ought to look like. For a point (x, y) that is on the curve, explain why $\sqrt{y^2} = \sqrt{x^2 + (y-2)^2}$. Square both sides of this equation and solve for y. Identify the curve.

66. Find an equation describing all points equidistant from the x-axis and $(1, 4)$ (see exercise 65).

EXPLORATORY EXERCISES

1. Suppose that a graphing calculator is set up with pixels corresponding to $x = 0, 0.1, 0.2, 0.3, \ldots, 2.0$ and $y = 0, 0.1, 0.2, 0.3, \ldots, 4.0$. For the function $f(x) = x^2$, compute the indicated function values and round off to give pixel coordinates [e.g., the point $(1.19, 1.4161)$ has pixel coordinates $(1.2, 1.4)$].

(a) $f(0.4)$, (b) $f(0.39)$, (c) $f(1.17)$, (d) $f(1.20)$, (e) $f(1.8)$, (f) $f(1.81)$. Repeat (c)–(d) if the graphing window is zoomed in so that $x = 1.00, 1.01, \ldots, 1.20$ and $y = 1.30, 1.31, \ldots, 1.50$. Repeat (e)–(f) if the graphing window is zoomed in so that $x = 1.800, 1.801, \ldots, 1.820$ and $y = 3.200, 3.205, \ldots, 3.300$.

2. Graph $y = x^2 - 1$, $y = x^2 + x - 1$, $y = x^2 + 2x - 1$, $y = x^2 - x - 1$, $y = x^2 - 2x - 1$ and other functions of the form $y = x^2 + cx - 1$. Describe the effect(s) a change in c has on the graph.

3. Figures 0.31 and 0.32 provide a catalog of the possible types of graphs of cubic polynomials. In this exercise, you will compile a catalog of graphs of fourth-order polynomials (i.e., $y = ax^4 + bx^3 + cx^2 + dx + e$). Start by using your calculator or computer to sketch graphs with different values of a, b, c, d and e. Try $y = x^4$, $y = 2x^4$, $y = -2x^4$, $y = x^4 + x^3$, $y = x^4 + 2x^3$, $y = x^4 - 2x^3$, $y = x^4 + x^2$, $y = x^4 - x^2$, $y = x^4 - 2x^2$, $y = x^4 + x$, $y = x^4 - x$ and so on. Try to determine what effect each constant has.

0.3 INVERSE FUNCTIONS

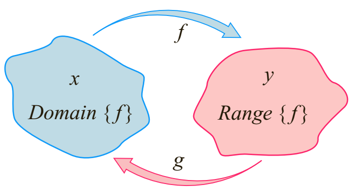

FIGURE 0.41
$g(x) = f^{-1}(x)$

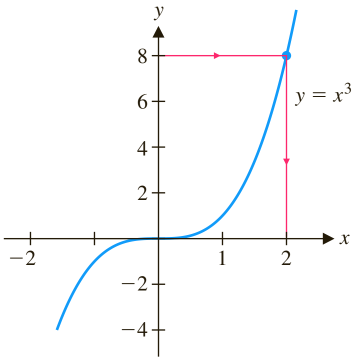

FIGURE 0.42
Finding the x-value corresponding to $y = 8$

The notion of an *inverse* relationship is basic to many areas of science. The number of common inverse problems is immense. As only one example, take the case of the electrocardiogram (EKG). In an EKG, technicians connect a series of electrodes to a patient's chest and use measurements of electrical activity on the surface of the body to infer something about the electrical activity on the surface of the heart. This is referred to as an *inverse* problem, since physicians are attempting to determine what *inputs* (i.e., the electrical activity on the surface of the heart) cause an observed *output* (the measured electrical activity on the surface of the chest).

The mathematical notion of inverse is much the same as that just described. Given an output (in this case, a value in the range of a given function), we wish to find the input (the value in the domain) that produced that output. That is, given a $y \in Range\{f\}$, find the $x \in Domain\{f\}$ for which $y = f(x)$. (See the illustration of the inverse function g shown in Figure 0.41.)

For instance, suppose that $f(x) = x^3$ and $y = 8$. Can you find an x such that $x^3 = 8$? That is, can you find the x-value corresponding to $y = 8$? (See Figure 0.42.) Of course, the solution of this particular equation is $x = \sqrt[3]{8} = 2$. In general, if $x^3 = y$, then $x = \sqrt[3]{y}$. In light of this, we say that the cube root function is the *inverse* of $f(x) = x^3$.

EXAMPLE 3.1 Two Functions That Reverse the Action of Each Other

If $f(x) = x^3$ and $g(x) = x^{1/3}$, show that

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x,$$

for all x.

Solution For all real numbers x, we have

$$f(g(x)) = f(x^{1/3}) = (x^{1/3})^3 = x$$

and

$$g(f(x)) = g(x^3) = (x^3)^{1/3} = x. \blacksquare$$

Notice in example 3.1 that the action of f undoes the action of g and vice versa. We take this as the definition of an inverse function. (Again, think of Figure 0.41.)